Ch-12

Legendre's equation and Legendre's function

Art-1 The differential equation of the form.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is called legendre's differential Equation.
Where n is a constant.

This equation can also be written as -

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

Art-2 Solution

The legendre's differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{——} (1)$$

It can be solved in series of ascending or descending powers of x. The Solution in descending powers of x is more Important than one in ascending.

Let $y = \sum_{r=0}^{\infty} a_r x^{m-r}$, be solution of ①

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m-r) x^{m-r-1}$$

and
$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2}$$

substituting in ①

$$(1-x^2) \sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2} - 2x \sum_{r=0}^{\infty} a_r (m-r) x^{m-r-1}$$
$$+ n(n+1) \sum_{r=0}^{\infty} a_r x^{m-r} = 0.$$

$$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2} - \sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^2 \cdot x^{m-r-2}$$
$$-2 \sum_{r=0}^{\infty} a_r (m-r) x^{m-r-1} \cdot x$$
$$+ n(n+1) \sum_{r=0}^{\infty} a_r x^{m-r} = 0$$

$$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1)x^{m-r-2} - \sum_{r=0}^{\infty} a_r (m-r)(m-r-1)x^{m-r}$$

$$-2\sum_{r=0}^{\infty} a_r (m-r)x^{m-r} + n(n+1)\sum_{r=0}^{\infty} a_r x^{m-r} = 0$$

$$\sum_{r=0}^{\infty} a_r \Big[(m-r)(m-r-1)x^{m-r-2} + \{n(n+1) - (m-r)(m-r-1) - 2(m-r)\}x^{m-r}\Big] = 0$$

$$\sum_{r=0}^{\infty} a_r \Big[(m-r)(m-r-1)x^{m-r-2} + \{n^2+n - (m-r)^2 + (m-r) - 2(m-r)\}x^{m-r}\Big] = 0$$

$$\sum_{r=0}^{\infty} a_r \Big[(m-r)(m-r-1)x^{m-r-2} + \{n^2+n - (m-r)^2 - (m-r)\}x^{m-r}\Big] = 0$$

$$\sum_{r=0}^{\infty} a_r \Big[(m-r)(m-r-1)x^{m-r-2} + \{n^2 - (m-r)^2 + n - (m-r)\}x^{m-r}\Big] = 0$$

$$\sum_{r=0}^{\infty} a_r \Big[(m-r)(m-r-1)x^{m-r-2} + \{(n-(m-r))(n+(m-r)) + (n-(m-r))\}x^{m-r}\Big] = 0$$

$$\sum_{r=0}^{\infty} a_r \left[(m-r)(m-r-1)x^{m-r-2} + (n-(m-r))(n+m-r+1)x^{m-r}\right]=0$$

$$\sum_{r=0}^{\infty} a_r \left[(m-r)(m-r-1)x^{m-r-2} + (n-m+r)(n+m-r+1)x^{m-r}\right]=0 \quad \text{——— (2)}$$

Now (2) being an identity, we can equate to zero the coeff. of various powers of x.

equating to zero the coeff. of highest power of x

i.e x^m we get.

$$a_0 (n-m)(n+m+1) = 0.$$

Now $a_0 \neq 0$.

$$\therefore (n-m) = 0$$

$$\left.\begin{array}{l} n = m \\ n+m+1 = 0 \\ m = -(n+1) \end{array}\right\} \text{——— (3)}$$

equating to zero coeff. of x^{m-1}

$$a_1 (n-m+1)(n+m) = 0.$$

$\therefore a_1 = 0$ since neither $(n-m+1)$ nor $(n+m)$ is zero.

by virtue of (3)

Again equating to zero the coefficient of the general term i.e x^{m-r}, we get

$$a_{r-2}(m-r+2)(m-r+1) + (n-m+r)(n+m-r+1)a_r = 0$$

$$a_r = -\frac{(m-r+2)(m-r+1)}{(n-m+r)(n+m-r+1)} a_{r-2} \quad (4)$$

Putting $r = 3$

$$a_3 = \frac{-(m-1)(m-2)}{(n-m+3)(n+m-2)} a_1 = 0$$

Since $a_1 = 0$.

we have $a_1 = a_3 = a_5 = \text{-------} = 0$ (each)

Case I when $m = n$

from (4) we have

$$a_r = \frac{-(n-r+2)(n-r+1)}{r \cdot (2n-r+1)} a_{r-2}$$

Putting $r = 2, 4 \ldots$ etc.

$$a_2 = \frac{-n(n-1)}{2(2n-1)} a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)} a_2$$

$$y = [a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \ldots]$$

$$y = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} \ldots \right] \quad \text{— (5)}$$

Which is one soln of Legendre's equation.

case II when $m = -(n+1)$

$$a_r = \frac{(n+r-1)(n+r)}{r(2n+r+1)} a_{r-2}$$

Putting $r = 2, 4 \ldots$

$$a_2 = \frac{(n+1)(n+2)}{2 \cdot (2n+3)} a_0$$

$$a_4 = \frac{(n+3)(n+4)}{4.(2n+5)}\, a_2$$

$$y = \sum_{r=0}^{\infty} a_r\, x^{-n-1-r} = a_0 x^{-n-1} + a_2 x^{-n-3} + a_4 x^{-n-5} + \cdots$$

$$= a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \cdots \right] \quad \text{—— (6)}$$

which is other solution of Legendre's equation.